Name: ____ E-mail ID: ________@virginia.edu

Pledge: ____

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Signature: ____

There are 180 minutes (3 hours) for this exam and 180 points on the test; don't spend too long on any one question! There is an exam reference sheet as the last page of this exam – feel free to tear it off. You may use scrap paper. However, all answers must be on these twelve exam pages (nothing on the exam reference sheet will be graded).

Part I: Logic	_____/20
Part II: Proofs	_____/50
Part III: Counting	_____/50
Part IV: Structures	_____/55
Part V: Applications	_____/10
Total	_____/180

Part I: Logic

1. (5 points) What is the converse and inverse of $p \to q$? Clearly label which is which!

2. (15 points) Using logical equivalences, prove that $(\neg p \wedge q) \to \left(\neg(q \to p)\right)$ is a tautology. In other words, show that $(\neg p \wedge q) \to \left(\neg(q \to p)\right) \equiv T$. You must clearly label each step!

Part II: Proofs

3. (5 points) Give an English statement that can be proven trivially (i.e. by a trivial proof).

4. (5 points) Give an English statement that can be proven vacuously (i.e. by a vacuous proof).

5. (10 points) Prove, via a proof by contradiction, that if n is an integer, and $3n+2$ is even, then n is even.

6. (15 points) Given the three propositions $\neg p$, $(\neg q \vee p)$, $\neg r \vee q$, can we conclude r? Show this by using rules of inference. You must label all your steps!

7. (15 points) Using weak mathematical induction, show that $1+2+2^2+\cdots+2^n = 2^{n+1}-1$ for all $n \geq 0$. You must label all your steps!

Part III: Counting

8. (5 points) How many functions are there from the set {1, 2, …, n}, where n is a positive integer, to the set {0, 1}?

9. (5 points) What is the coefficient of x^7 in $(2x+1)^{13}$? Leave your answer in combinatorial form.

10. (5 points) What must be shown in a combinatorial proof? In other words, what must you do in order to prove a formula via a combinatorial proof?

11. (15 points) What is the probability of being dealt a straight in a 5-card poker hand? Leave your answer in combinatorial form. Recall that a straight is a series of 5 cards in a sequence. For example, A, 2, 3, 4, 5 is a straight, as is 10, J, Q, K, A (note that the ace can be high or low). Suit does not matter in a straight (and we are ignoring straight flushes and royal flushes).

12. (10 points) Let $X = \{x_1, x_2, \ldots, x_{100}\}$ be a set of 100 distinct positive integers. If these positive integers are divided by 75, then show that at least two of the remainders must be the same.

13. (10 points) From the set of integers in the set {1, 2, …, 30}, what is the least number of integers that must be chosen so that at least one of them is divisible by either 3 or 5? Explain your answer.

Part IV: Structures

14. (5 points) What is the cardinality of a power set of a set of *n* elements? (déjà vu…)

15. (5 points) Describe, in English, what 1-to-1 and onto mean for functions.

16. (5 points) What is the difference between a sequence that has an arithmetic progression, and one that has a geometric progression?

17. (5 points) Describe, in English, what transitive closure means.

18. (5 points) What is the difference between asymmetry and antisymmetry?

19. (5 points) Which properties are required for an equivalence relation and which are required for a partial ordering?

20. (10 points) Given a relation R, what is the difference between R^*, R^{-1}, and $\overline{R}$? Clearly describe what each means.

21. (15 points) Given sets A and B, prove that $(A-B)\bigcup(B-A)=(A\bigcup B)-(B\bigcap A)$. You can either use set builder notation or set equivalences, but you cannot use membership tables.

Part V: Applications

22. (10 points) State a real world application of each of the following discrete mathematical concepts.

- Boolean logic
- Mathematical induction
- Relations
- Prime numbers
- Algorithms
- Fibonacci sequence

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Final Exam Reference Sheet

Set and logical identities

Sets (Rosen, p. 89)	Name	Boolean logic (Rosen, p. 24)
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws	$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws	$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$
$A \cup A = A$ $A \cap A = A$	Idempotent laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
$\left(\overline{\overline{A}}\right) = A$	Complementation law	$\neg(\neg p) \equiv p$
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	DeMorgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws	$p \vee \neg p = \mathbf{T}$ $p \wedge \neg p = \mathbf{F}$

Rules of inference (Rosen, p. 58)

Rule of Inference	Tautology	Name
$\underline{p}$ $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
p $\underline{q}$ $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
p $\underline{p \rightarrow q}$ $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$\neg q$ $\underline{p \rightarrow q}$ $\therefore \neg p$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \vee q$ $\underline{\neg p}$ $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Disjunctive syllogism
$p \vee q$ $\underline{\neg p \vee r}$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

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Do not include any work on this page.
It will not be graded